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On Some Complete Tripartite Graphs that Decline Continuous Monotonic Decomposition

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***Abstract* - A collection of complete tripartite graphs, viz., $K_{1,3,m}$, $K_{2,3m}$, $K_{2,5m}$ and $K_{3,5,m}$ do not accept Continuous Monotonic Decomposition(CMD). It is shown that by an addition or removal of a single edge will make these graphs accept CMD. Eventually, the discussion helps to find a series of numbers which are not triangular.**

***Keywords :* Graph Decomposition, Extremal Graphs, Complete Tripartite Graphs, Continuous Monotonic Decomposition, Triangular Numbers**

1. Introduction

An undirected simple graph is an ordered pair $G = (\mathcal{V}, \mathcal{E})$ comprising a set $\mathcal{V}$ of vertices together with a set $\mathcal{E}$ of edges, which are 2-element subsets of $\mathcal{V}$ (i.e., an edge is related with two vertices, and the relation is represented as unordered pair of the vertices with respect to the particular edge). A path in a graph is a sequence of vertices such that from each of its vertices there is an edge to the next vertex in the sequence. Graph with the property that there is a path between every pair of vertices is known as a connected graph. A graph G, referred to here is an undirected connected simple graph.

A complete m-partite graph $G = K_{n1, n2,.....nm}$ $\forall$ $n_1, n_2, \ldots n_m \in \mathbb{N}$ is a graph whose vertex set $\mathcal{V}$ can be partitioned into m subsets $\mathcal{V}_1, \mathcal{V}_2, \ldots, \mathcal{V}_m$ such that every edge of G joins every vertex of $\mathcal{V}_i$ with every vertex of $\mathcal{V}_j$ where $i \neq j$ and $|\mathcal{V}_i| = n_i$, i= 1 to m. When m=2, G is a complete bipartite graph and m=3, G is a complete tripartite graph.

II. Graph Decompositions

Let $G=(\mathcal{V}, \mathcal{E})$ be a connected simple graph. If $H_1, H_2, \ldots H_k$ $\forall$ $k \in \mathbb{N}$ are edge-disjoint subgraphs of $G \ni \mathcal{E}(G) = \mathcal{E}(H_1) \cup \mathcal{E}(H_2) \cup \ldots \cup \mathcal{E}(H_k)$, then $H_1, H_2, \ldots H_k$ is said to be a decomposition of G. Different types of decomposition of G have been studied in the literature by imposing suitable conditions on the subgraphs H_i.

Gnana Dhas and Paulraj Joseph introduced the concept known as continuous monotonic decomposition of graphs [2]. A decomposition, $H_1, H_2, \ldots H_k$ $\forall$ $k \in \mathbb{N}$, is said to be a Continuous Monotonic Decomposition (CMD) if each H_i is a connected subraph and $|\mathcal{E}(H_i)| = i$ $\forall$ $i \in \mathbb{N}$.

Example 1

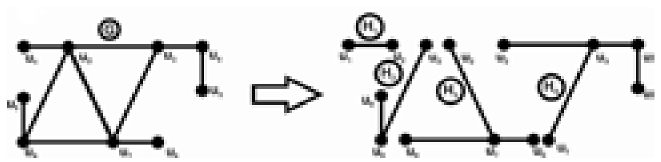

Fig. 1 Continuous Monotonic Decomposition of G into H_1, H_2, H_3, and H_4

III. Some Number Theory Concepts

Triangular number is a natural number that is the sum consecutive natural numbers, beginning with 1. Pythagoras found that a number is triangular if and only if it is of the form $\frac{n(n+1)}{2}$ for some $n \geq 1$. Plutarch stated that n is a triangular number if and only if $8n+1$ is a perfect square. The square of any integer is either of the form 3k or 3k+1 for some $k \in \mathbb{N}$.

Euler identified that if n is a triangular number, then so are $9n+1$, $25n+3$ and $49n+6$. If t_n denotes the n^{th} triangular number, then $t_n = {}^{(n+1)}C_2$. All these number theory results are used in the sense of David M. Burton [5].

IV. Continuous Monotonic Decomposition of Graphs

Continuous Monotonic Decomposition of a wide variety of graphs had been studied by Gnana Dhas and Paulraj Joseph, and Navaneetha Krishnan and Nagarajan [3]-[5]. If a graph G admits a CMD of $H_1, H_2, \ldots H_k$ $\forall$ $k \in \mathbb{N}$ if and only if $q = {}^{(n+1)}C_2$ [3]. But we know that for any positive integer n, ${}^{(n+1)}C_2$ is a triangular number. Hence, if we are able to find out the number of the edges of any connected graph, it is easy for us to conclude whether it admits CMD or not. Joseph Varghese and A. Antonysamy discussed on various types of Continuous Monotonic Decomposition of Complete Tripartite Graphs and product of graphs [6]-[8].

In this paper, a collection of complete tripartite graphs is enlisted which are not admitting Continuous Monotonic Decomposition. It is also found that if we slightly modify

certain complete tripartite graphs, the modified graphs accept Continuous Monotonic Decomposition. Eventually, the discussion leads to finding a sequence of non-triangular numbers.

We start with $K_{m,m+k,m+2k}$

Theorem 1: A complete tripartite graph $K_{m,m+k,m+2k}$ does not accept Continuous Monotonic Decomposition $\forall$ m and $k \in \mathbb{N}$ when k is not a multiple of 3.

Proof: Let q(G) denote the size of a graph G.

We know that a graph G accepts CMD of $H_1, H_2, \ldots H_n$ if and only if $q(G) = \frac{n(n+1)}{2}$ $\forall$ $n \in \mathbb{N}$……..(1).

But $\frac{n(n+1)}{2}$ is a triangular number $\forall$ $n \in \mathbb{N}$.

Now, the given graph is $K_{m,m+k,m+2k}$.

$$\therefore q(G) = \frac{[m(m+k+m+2k)+(m+k)(m+m+2k)+(m+2k)(m+k+m)]}{2}$$

i.e., $=3m^2+6mk+2k^2$……..(2)

Hence, $K_{m,m+k,m+2k}$ to accept CMD, $3m^2+6mk+2k^2$ should be a triangular number $\forall$ m, $k \in \mathbb{N}$.

By the property of triangular numbers, we know that if n is a triangular number, then $8n+1$ is a perfect square.

Hence, $8(3m^2+6mk+2k^2)+1$ should be a perfect square, in order that $K_{m,m+k,m+2k}$ accepts CMD.

Also we know from elementary number theory that if n is the square of an integer, then $n \equiv 0$ or 1 mod(3). i.e., if $n \equiv 2$ mod(3), then n is not a perfect square.

i.e., if $8(3m^2+6mk+2k^2)+1 \equiv 2$ mod(3), then $K_{m,m+k,m+2k}$ does not accept a CMD.

i.e., if $16k^2+1 \equiv 2$ mod(3), then $K_{m,m+k,m+2k}$ does not accept a CMD.

But, when $k \equiv 1$ or 2 mod(3), then $k^2 \equiv 1$ mod(3), and $16k^2+1 \equiv 2$ mod(3).

$\therefore$ if k is not a multiple of 3, then $16k^2+1 \equiv 2$ mod(3).

i.e., if k is not a multiple of 3, then $8(3m^2+6mk+2k^2)+1 \equiv 2$ mod(3).

i.e., if k is not a multiple of 3, then $8(3m^2+6mk+2k^2)+1$ is not a perfect square.

i.e., if k is not a multiple of 3, then $3m^2+6mk+2k^2$ is not a triangular number.

i.e., if k is not a multiple of 3, then $K_{m,m+k,m+2k}$ does not accept CMD.

Hence, a complete tripartite graph $K_{m,m+k,m+2k}$ does not accept Continuous Monotonic Decomposition $\forall$ m and $k \in \mathbb{N}$, when k is not a multiple of 3.

Example 2

When m=1, k=1, we have $K_{1,2,3}$

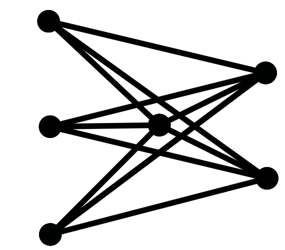

Fig. 2 $K_{1,2,3}$ does not accept CMD

Naturally, $K_{1,2,3}$ does not accept CMD because it is a complete tripartite graph $K_{m,m+k,m+2k}$ with m=1 and k=1 and k is not a multiple of 3. Also we can verify that $q(K_{m,m+k,m+2k})$ is 11 and it is not a triangular number.

But it is not just that. $K_{1,2,n}$ does not accept CMD $\forall$ $n \in \mathbb{N}$!

Let us verify this interesting result.

We have, $q(K_{1,2,m}) = \frac{[m(1+2)+1(m+2)+2(m+1)]}{2}$

i.e., $= 3m+2$, $\forall$ $m \in \mathbb{N}$…..(3)

We know that G accepts CMD $H_1, H_2, \ldots, H_n$ if and only if $q(G) = \frac{n(n+1)}{2}$ $\forall$ $n \in \mathbb{N}$…….(4)

Now, (4) becomes, 3m+2.

But, then 3m+2 has to be a triangular number.

i.e., 8(3m+2)+1 should be a perfect square….(Property of triangular numbers)

i.e., 24m+16+1 should be a perfect square.

But, $24m+17 \equiv 2$ mod(3) and hence it cannot be a perfect square. (Property of Perfect Squares).

This implies that $q(K_{1,2,n})$ can never be a triangular number for any $n \in \mathbb{N}$.

This is stated as,

Theorem 2: A complete tripartite graph $K_{1,2,n}$ does not accept a Continuous Monotonic Decomposition, $\forall$ $n \in \mathbb{N}$.

Definition 1: A fan graph $F_{m,n}$ is defined as the graph join $\overline{K}_m + P_n$ where $\overline{K}_m$ is the empty graph on m nodes and P_n is the path graph on n nodes. The $(r,3)$-fan graph is isomorphic to the complete tripartite graph $K_{1,2,r}$.

Example 3

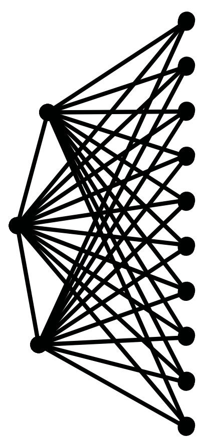

Fig. 3 $K_{1,2,10}$ does not accept CMD

Though $K_{1,2,n}$ does not accept CMD, removal of an edge or addition of an edge makes $K_{1,2,n}$ eligible for CMD for some values of n. Next two results are about this.

Theorem 3: A complete tripartite graph $K_{1,2,m}-\{e\}$ accepts Continuous Monotonic Decomposition of $H_1, H_2, \ldots, H_{3n+1}$ if and only if $m=(3n^2+3n)/2\ \forall\ n\in\mathbb{N}$.

Proof: Assume that a complete tripartite graph $K_{1,2,m}-\{e\}$, accepts CMD of $\{H_1, H_2, \ldots, H_{3n+1}\}\ \forall\ n\in\mathbb{N}$.

We have, $q(K_{1,2,m}-\{e\}) = (3m+2)-1$ [Using (3)]

$= 3m+1\ \forall\ m\in\mathbb{N}$…..(5)

We know that G accepts CMD $H_1, H_2, \ldots, H_n$ if and only if $q(G)=\frac{n(n+1)}{2}\ \forall\ n\in\mathbb{N}$.

i.e., graph $K_{1,2,m}-\{e\}$ accepts CMD $H_1, H_2, \ldots, H_{3n+1}$ if and only if $q(K_{1,2,m}-\{e\}) = \frac{(3n+1)(3n+2)}{2}\ \forall\ n\in\mathbb{N}$.

i.e., $q(K_{1,2,m}-\{e\})$ must be a member of the sequence 1, 3, 6, 10, 15, 21…..,$\frac{k(k+1)}{2}\ \forall\ k\in\mathbb{N}$…….. (6)

i.e., $\frac{(3n+1)(3n+2)}{2} = \frac{k(k+1)}{2}$ for some $k\in\mathbb{N}$ and $\forall\ n\in\mathbb{N}$.

i.e., $k = 3n+1\ \forall\ n\in\mathbb{N}$….(7)

Also, $K_{1,2,m}-\{e\}$ accepts CMD if and only if $q(K_{1,2,m}-\{e\})$ is one among the members of the sequence given in (6).

i.e., 3m+1 should be one of these values….. [Using (5)]

i.e., $3m+1 = \frac{k(k+1)}{2}$ for some $k\in\mathbb{N}$.

i.e., $3m+1 = \frac{(3n+1)(3n+2)}{2}$….[using (7)]

i.e., $3m = (9n^2+9n)/2\ \forall\ n\in\mathbb{N}$.

i.e., $m = (3n^2+3n)/2\ \forall\ n\in\mathbb{N}$.

The first few values of m are 3, 9, 18, 30, 45, 63, 84….

Conversely,

Suppose that $K_{1,2,m}$ is a complete tripartite graph with $m=(3n^2+3n)/2\ \forall\ n\in\mathbb{N}$.

We know that $q(K_{1,2,m}) = 3m+2$

i.e., when $m = (3n^2+3n)/2$,

$$\begin{aligned} q(K_{1,2,m}) &= ((9n^2+9n)/2)+2 \\ &= ((9n^2+9n+4)/2) \\ &= ((3n+1)(3n+2)+2)/2) \\ &= ((3n+1)(3n+2)/2)+1 \ldots..(8) \end{aligned}$$

(8) is of the form $[k(k+1)/2]+1,\ \forall\ k\in\mathbb{N}$.

$\therefore$ removing one edge from $K_{1,2,m}$, we get $q((K_{1,2,m})-\{e\}) = ((3n+1)(3n+2)/2)$…[Using (8)]

This implies that $K_{1,2,m}-\{e\}$ being a connected simple graph, can be decomposed into $H_1, H_2, \ldots H_k\ \forall\ k\in\mathbb{N}$.

i.e., when $m=(3n^2+3n)/2$, $K_{1,2,m}-\{e\}$ can be decomposed into $H_1, H_2, \ldots H_{3n+1}\ \forall\ n\in\mathbb{N}$.

Example 4

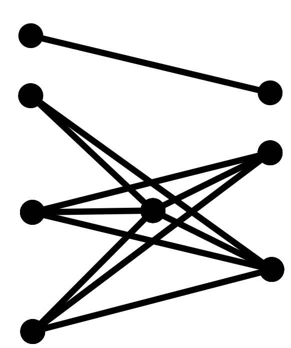

Fig. 4 $K_{1,2,3}-\{e\}$ accepts CMD

TABLE I LIST OF FIRST 25 $K_{1,2,M}$'S WHICH ACCEPT CMD IF AN EDGE IS REMOVED

m	$q((K_{1,2,m}))$	$q((K_{1,2,m})-\{e\})$	CMD
3	11	10	$H_1, H_2, \ldots H_4$
9	29	28	$H_1, H_2, \ldots H_7$
18	56	55	$H_1, H_2, \ldots H_{10}$
30	92	91	$H_1, H_2, \ldots H_{13}$
45	137	136	$H_1, H_2, \ldots H_{16}$
63	191	190	$H_1, H_2, \ldots H_{19}$
84	254	253	$H_1, H_2, \ldots H_{22}$
108	326	325	$H_1, H_2, \ldots H_{25}$
135	407	406	$H_1, H_2, \ldots H_{28}$
165	497	496	$H_1, H_2, \ldots H_{31}$
198	596	595	$H_1, H_2, \ldots H_{34}$
234	704	703	$H_1, H_2, \ldots H_{37}$
273	821	820	$H_1, H_2, \ldots H_{40}$
315	947	946	$H_1, H_2, \ldots H_{43}$
360	1082	1081	$H_1, H_2, \ldots H_{46}$
408	1226	1225	$H_1, H_2, \ldots H_{49}$
459	1379	1378	$H_1, H_2, \ldots H_{52}$
513	1541	1540	$H_1, H_2, \ldots H_{55}$
570	1712	1711	$H_1, H_2, \ldots H_{58}$
630	1892	1891	$H_1, H_2, \ldots H_{61}$
693	2081	2080	$H_1, H_2, \ldots H_{64}$
759	2279	2278	$H_1, H_2, \ldots H_{67}$
828	2486	2485	$H_1, H_2, \ldots H_{70}$
900	2702	2701	$H_1, H_2, \ldots H_{73}$
975	2927	2926	$H_1, H_2, \ldots H_{76}$

Theorem 4: A complete tripartite graph $K_{1,2,m}\cup\{e\}$ accepts Continuous Monotonic Decomposition of $H_1, H_2, \ldots, H_{3n+2}$ and $H_1, H_2, \ldots, H_{3n+3}$ if and only if $m=(3n^2+5n)/2$ and $(3n^2+7n+2)/2$ respectively $\forall\ n\in\mathbb{N}$.

Proof : We have, $q(K_{1,2,m}\cup\{e\}) = (3m+2)+1\ \forall\ m\in\mathbb{N}$. [Using (3)]

i.e., $=3m+3$…….(9)

We know that G accepts CMD $\{H_1, H_2, \ldots, H_n\}$ if and only if $q(G)= n(n+1)/2,\ \forall\ n\in\mathbb{N}$.

Case 1:

Assume that a complete tripartite graph $K_{1,2,m}\cup\{e\}$, accepts CMD of $H_1, H_2, \ldots, H_{3n+2}\ \forall\ n\in\mathbb{N}$.

i.e., graph $K_{1,2,m}\cup\{e\}$ accepts CMD $H_1, H_2, \ldots, H_{3n+2}$ if and only if $q(K_{1,2,m}\cup\{e\}) = (3n+2)(3n+3)/2\ \forall\ n\in\mathbb{N}$.

i.e., $q(K_{1,2,m}\cup\{e\})$ must be a member of the sequence 1, 3, 6, 10, 15, 21….., $k(k+1)/2\ \forall\ k\in\mathbb{N}$. (10)

i.e., $(3n+2)(3n+3)/2= k(k+1)/2$ for some $k\in\mathbb{N}$ and $\forall\ n\in\mathbb{N}$.

i.e., $k = 3n+2,\ \forall\ n\in\mathbb{N}$….(11)

Also, $K_{1,2,m}\cup\{e\}$ accepts CMD if and only if $q(K_{1,2,m}\cup\{e\})$ is one among the members of the sequence given in (10).

i.e., 3m+3 should be one of these values…[Using (9)]

i.e., $3m+3 = k(k+1)/2$ for some $k\in\mathbb{N}$.

i.e., $3m+3 =(3n+2)(3n+3)/2$….[Using (11)]

i.e., $3m =(9n^2+15n)/2\ \forall\ n\in\mathbb{N}$.

i.e., $m =(3n^2+5n)/2\ \forall\ n\in\mathbb{N}$.

The first few values of m are 4, 11, 21, 34, 50…….

Example 5

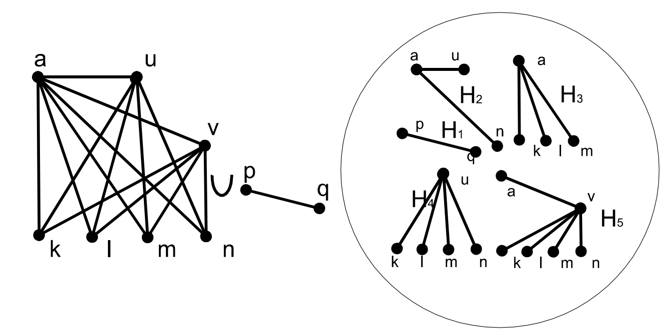

Fig. 5 $K_{1,2,4}\cup(p,q)$ and its CMD

Case 1:

Assume that a complete tripartite graph $K_{1,2,m}\cup\{e\}$, accepts CMD of $H_1, H_2, \ldots, H_{3n+3}\ \forall\ n\in\mathbb{N}$.

i.e., graph $K_{1,2,m}\cup\{e\}$ accepts CMD $H_1, H_2, \ldots, H_{3n+3}$ if and only if $q(K_{1,2,m}\cup\{e\}) = (3n+3)(3n+4)/2\ \forall\ n\in\mathbb{N}$.

i.e., $(3n+3)(3n+4)/2= k(k+1)/2$ for some $k\in\mathbb{N}$ and $\forall\ n\in\mathbb{N}$.

i.e., $k = 3n+3\ \forall\ n\in\mathbb{N}$….(12)

Also, $K_{1,2,m}\cup\{e\}$ accepts CMD if and only if $q(K_{1,2,m}\cup\{e\})$ is one among the members of the sequence given in (10).

i.e., 3m+3 should be one of these values...[Using (9)]

i.e., $3m+3 = k(k+1)/2$ for some $k\in\mathbb{N}$.

i.e., $3m+3 = (3n+3)(3n+4)/2$….[Using (12)]

i.e., $3m = (9n^2+21n+6)/2\ \forall\ n\in\mathbb{N}$.

i.e., $m = (3n^2+7n+2)/2\ \forall\ n\in\mathbb{N}$.

The first few values of m are 6, 14, 25, 39…….

Example 6

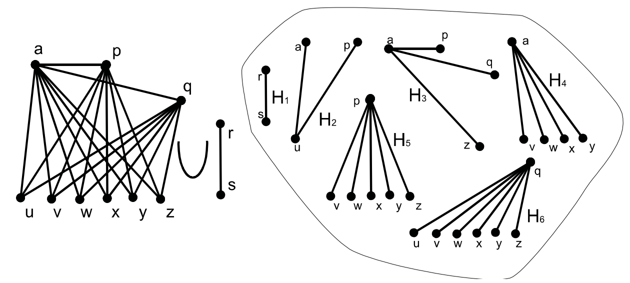

Fig. 6 $K_{1,2,6}\cup(r,s)$ and its CMD

Conversely,

Suppose that $K_{1,2,m}$ is a complete tripartite graph with $m=(3n^2+5n)/2$ or $m = (3n^2+7n+2)/2\ \forall\ n\in \mathbb{N}$.

We know that $q(K_{1,2,m}) = 3m+2$

Case 1: when $m =(3n^2+5n)/2$

$$\begin{aligned} q(K_{1,2,m}) &= 3m+2 \\ &= ((9n^2+15n)/2)+2 \\ &= ((9n^2+15n+4)/2) \\ &= ((3n+2)(3n+3)-2)/2) \\ &= ((3n+2)(3n+3)/2)-1 \quad \ldots(13) \end{aligned}$$

(13) is of the form $[k(k+1)/2]-1,\ \forall\ k \in \mathbb{N}$.

∴ connecting an edge to $K_{1,2,m}$, we get $q((K_{1,2,m})\cup\{e\}) = ((3n+2)(3n+3)/2)$.

This implies that $K_{1,2,m}\cup\{e\}$ being a connected simple graph, can be decomposed into $\{H_1, H_2, .. H_k\}$ for some $k \in \mathbb{N}$.

i.e., when $m=(3n^2+5n)/2$, $K_{1,2,m}\cup\{e\}$ can be decomposed into $H_1, H_2, \ldots H_{3n+2}\ \forall\ n \in \mathbb{N}$.

Case 2: when $m =(3n^2+7n+2)/2$

$$\begin{aligned} q(K_{1,2,m}) &= 3m+2 \\ &= ((9n^2+21n+6)/2)+2 \\ &= ((9n^2+21n+6)/2)+2 \\ &= ((9n^2+21n+10)/2) \\ &= ((3n+3)(3n+4)-2)/2) \\ &= ((3n+3)(3n+4)/2)-1 \quad \ldots(14) \end{aligned}$$

(14) is of the form $[k(k+1)/2]-1,\ \forall\ k \in \mathbb{N}$.

∴ connecting an edge to $K_{1,2,m}$, we get $q((K_{1,2,m})\cup\{e\}) = ((3n+3)(3n+4)/2)$.

This implies that $K_{1,2,m}\cup\{e\}$ being a connected simple graph, can be decomposed into $H_1, H_2, \ldots H_k\ \forall\ k \in \mathbb{N}$.

i.e., when $m=(3n^2+7n+2)/2$, $K_{1,2,m}\cup\{e\}$ can be decomposed into $H_1, H_2, \ldots H_{3n+3}\ \forall\ n \in \mathbb{N}$.

TABLE II LIST OF FIRST 25 $K_{1,2,M}$'S WHICH ACCEPT CMD IF AN EDGE IS ADDED

m	$q((K_{1,2,m}))$	$q((K_{1,2,m})\cup\{e\})$	CMD
4	14	15	$H_1, H_2, \ldots H_5$
6	20	21	$H_1, H_2, \ldots H_6$
11	35	36	$H_1, H_2, \ldots H_8$
14	44	45	$H_1, H_2, \ldots H_9$
21	65	66	$H_1, H_2, \ldots H_{11}$
25	77	78	$H_1, H_2, \ldots H_{12}$
34	104	105	$H_1, H_2, \ldots H_{14}$
39	119	120	$H_1, H_2, \ldots H_{15}$
50	152	153	$H_1, H_2, \ldots H_{17}$
56	170	171	$H_1, H_2, \ldots H_{18}$
69	209	210	$H_1, H_2, \ldots H_{20}$
76	230	231	$H_1, H_2, \ldots H_{21}$
91	275	276	$H_1, H_2, \ldots H_{23}$
99	299	300	$H_1, H_2, \ldots H_{24}$
116	350	351	$H_1, H_2, \ldots H_{26}$
125	377	378	$H_1, H_2, \ldots H_{27}$
144	434	435	$H_1, H_2, \ldots H_{29}$
154	464	465	$H_1, H_2, \ldots H_{30}$
175	527	528	$H_1, H_2, \ldots H_{32}$
186	560	561	$H_1, H_2, \ldots H_{33}$
209	629	630	$H_1, H_2, \ldots H_{35}$
221	665	666	$H_1, H_2, \ldots H_{36}$
246	740	741	$H_1, H_2, \ldots H_{38}$
259	779	780	$H_1, H_2, \ldots H_{39}$
286	860	861	$H_1, H_2, \ldots H_{41}$

Here is another collection of complete tripartite graphs which do not accept CMD.

Theorem 5: A complete tripartite graph $K_{4,m,n}$ does not accept Continuous Monotonic Decomposition if either m or $n \equiv 2 \bmod(3)$, $\forall$ m, n$\in \mathbb{N}$.

Proof: We know that a graph G accepts CMD of $H_1, H_2, .., H_n$ if and only if $q(G)= n(n+1)/2$, $\forall\ n \in \mathbb{N}$.

Now, given graph is $K_{4,m,n.}$

$$\therefore q(K_{4,m,n}) = [4(m+n)+m(4+n) +n(4+m)]/2$$
$$= 4m+4n+mn \quad \ldots(15)$$

Given that, either m or $n \equiv 2 \bmod(3)$ and we have $q(K_{4,m,n}) = 4m+4n+mn$.

∴ for $K_{4,m,n}$ to accept CMD, $(4m+4n+mn)$ has to be a triangular number.

Now, $(4m+4n+mn)$ is a triangular number only if $8(4m+4n+mn)+1$ is a perfect square (Property of Triangular Numbers).

If $8(4m+4n+mn)+1$ is a perfect square, then $8(4m+4n+mn)+1\equiv 0$ or $1 \bmod(3)$.

i.e., if $8(4m+4n+mn)+1\equiv 2 \bmod(3)$, then $8(4m+4n+mn)+1$ is not a perfect square.

Case 1: $m \equiv 2 \bmod (3)$ and $n \equiv 0 \bmod (3)$.

Given that $m\equiv 2 \bmod(3)$ and $n\equiv 0 \bmod (3)$.

∴ $4m \equiv 2 \bmod (3)$, $4n \equiv 0 \bmod (3)$ and $mn \equiv 0 \bmod (3)$.

i.e., $4m+4n+mn \equiv 2 \bmod (3)$.

i.e., $8(4m+4n+mn)+1 \equiv 2 \bmod (3)$.

∴ $8(4m+4n+mn)+1$ is not a perfect square.

Case 2: $m \equiv 2 \bmod (3)$ and $n \equiv 1 \bmod (3)$.

Given that $m \equiv 2 \bmod(3)$ and $n \equiv 1 \bmod (3)$.

∴ $4m \equiv 2 \bmod (3)$, $4n \equiv 1 \bmod (3)$ and $mn \equiv 2 \bmod (3)$.

i.e., $4m+4n+mn \equiv 2 \bmod (3)$.

i.e., $8(4m+4n+mn)+1 \equiv 2 \bmod (3)$.

∴ $8(4m+4n+mn)+1$ is not a perfect square.

Case 3: $m \equiv 2 \bmod (3)$ and $n \equiv 2 \bmod (3)$.

Given that $m \equiv 2 \bmod(3)$ and $n \equiv 2 \bmod (3)$.

∴ $4m \equiv 2 \bmod (3)$, $4n \equiv 2 \bmod (3)$ and $mn \equiv 1 \bmod (3)$.

i.e., $4m+4n+mn \equiv 2 \bmod (3)$.

i.e., $8(4m+4n+mn)+1 \equiv 2 \bmod (3)$.

∴ $8(4m+4n+mn)+1$ is not a perfect square.

Case 4: $m \equiv 0 \bmod (3)$ and $n \equiv 2 \bmod (3)$

Similar as Case 1.

Case 5: $m \equiv 1 \bmod (3)$ and $n \equiv 2 \bmod (3)$.

Similar as Case 2.

So by analyzing all the cases above it is established that $\forall$ m, n $\in \mathbb{N}$, if either m or $n \equiv 2 \bmod (3)$, $(4m+4n+mn)$ is never a triangular number.

∴ a complete tripartite graph $K_{4,m,n}$ does not accept CMD if either m or $n \equiv 2 \bmod(3)$, $\forall$ m, n$\in \mathbb{N}$.

Example 7:

When m=2 n = 6, we have $K_{4,2,6}$

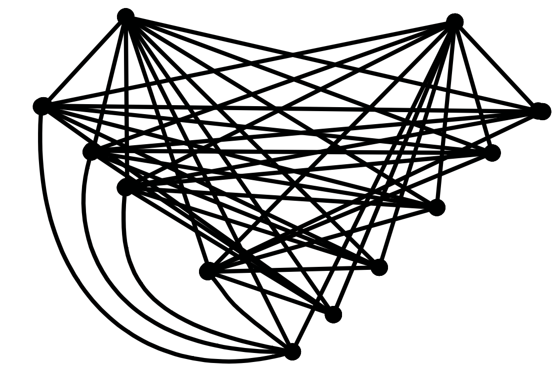

Fig 7 $K_{4,2,6}$ does not accept CMD

Theorem 6: A complete tripartite graph $K_{4,m,n}$ does not accept Continuous Monotonic Decomposition $\forall$ m, n$\in \mathbb{N}$ and m<4.

Proof: We know that a graph G accepts CMD of $\{H_1, H_2, .., H_n\}$ if and only if $q(G)= n(n+1)/2$, $\forall\ n \in \mathbb{N}$.

Now, given graph is $K_{4,m,n.}$

$$\therefore q(K_{4,m,n}) = [4(m+n)+m(4+n) +n(4+m)]/2$$
$$= 4m+4n+mn \quad \ldots(16)$$

Case 1: m=1

Now, (16) becomes $5n+4 \ldots(17)$

(1) is a triangular number $\forall$ n$\in \mathbb{N}$.

We know from elementary number theory that a triangular number will never end in 4 or 9.

But $\forall\ n \in \mathbb{N}$ (17) ends in 4 or 9 only. Hence (17) cannot be a triangular number. Hence, when m=1, $K_{m,4,n}$ does not accept CMD.

Case 2: m=2

(This case can be obtained if we put m=2, in Theorem 5). Now, (16) becomes, $6n+8 \ldots(18)$

(1) is a triangular number $\forall$ n$\in \mathbb{N}$.

But (18) is never a triangular number $\forall\ n \in \mathbb{N}$.

Because, if it is a triangular number then, $8(6n+8)+1$ is a perfect square. (Test for triangular numbers)

i.e., $48n+65$ is a square.

If 48n+65 is a square, then it is either 3k or 3k+1 for some k. (Property of Squares)

But, 48n+65 = 3(16n)+ 3(21)+2
i.e., = 3(16n+21)+2.
i.e., = 3k+2 for some k ∈ ℕ.
i.e., 48n+65 is not a perfect square ∀ n ∈ ℕ.
i.e., 6n+8 is not a triangular number.
Hence, when m=2, $K_{m,4,n}$ does not accept CMD.

Case 3: when m=3
Now, (16) becomes 7n+12.
i.e., 7n+12 = k(k+1)/2 for some k
i.e., 14n+24 = k^2+k for some k
i.e., k^2+k -(14n+24)=0.

i.e., $k = \frac{-1 \pm \sqrt{1+4(14n+24)}}{2}$

i.e., $= \frac{-1+\sqrt{56n+97}}{2}$ (Discarding the negative root)

But, for no value of n, $(\sqrt{56n+97}-1)$ is an integer

Hence, k is never an integer.

∴ when m=3, $K_{m,4,n}$ does not accept CMD.

Theorem 7: A complete tripartite graph $K_{1,5,n}$ does not accept a CMD, ∀ n∈ ℕ.

Proof: We have, $q(K_{1,5,m}) = \frac{[m(1+5)+1(m+5)+5(m+1)]}{2}$

= 6m+5, ∀ m ∈ ℕ…..(19)

We know that G accepts CMD $\{H_1, H_2, .., H_n\}$ if and only if q(G)= n(n+1)/2, ∀ n ∈ ℕ.

i.e., (19) should be of the form n(n+1)/2 for some n ∈ ℕ. i.e., 6m+5 should be a triangular number.

i.e., 8(6m+5)+1 should be a perfect square.

i.e., 48m+41 should be of the form 3k or 3k+1.

But, 48m+41=3k+2, for some k ∈ ℕ and ∀ m ∈ ℕ.

i.e., 48m+41 is not a perfect square and hence 6m+5 is not a triangular number.

Hence, a complete tripartite graph $K_{1,5,n}$ does not accept a CMD, ∀ n∈ ℕ.

Example 8

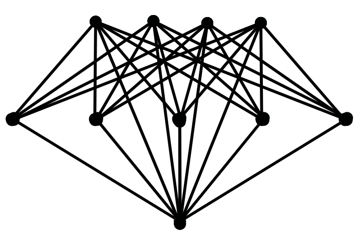

Fig 8 $K_{1,5,4}$ does not accept CMD

V. Conclusion

Continuous Monotonic Decomposition (CMD) is a special type of Ascending Subgraph Decomposition [1]. Since the size of the graph is n(n+1)/2, CMD is closely related to the theory of triangular numbers. The discussion in this paper leads to the identification of a collection of sequence of natural numbers that are not triangular. They are the following:

1. $3m^2+6mk+2k^2$ is not a triangular number ∀ m and k ∈ ℕ when k is not a multiple of 3.
2. 3m+2 is not a triangular number ∀ m ∈ ℕ.
3. 3m+1 is a triangular number if and only if m=3n(n+1)/2 ∀ n ∈ ℕ.
4. 3(m+1) is a triangular number if and only if m=n(3n+5)/2 or $(3n^2+7n+2)/2$ ∀ n ∈ ℕ.
5. 4(m+n)+mn is not a triangular number if either m or n ≡ 2 (mod 3) ∀ m, n ∈ ℕ.
6. 4(m+n)+mn is not a triangular number ∀ m, n ∈ ℕ with m<4.

References

[1] Y. Alavi, A. J. Boals, G. Chartrand, P. Erdös and O.R. Ollermann, "The Ascending Subgraph Decomposition Problem," *Congressus Numerantium,* Vol. 58, pp.7-14, 1987.

[2] N. G. Dhas and J. P. Joseph, "Continuous Monotonic Decomposition of Graphs," *International Journal of Management and Systems,* Vol 16, No. 3, pp. 333-344, Sept-Dec, 2000.

[3] N. G. Dhas and J. P. Joseph, "Continuous Monotonic Decomposition of Cycles," *International Journal of Management and Systems,* Vol 19, No. 1, pp. 65-76, Jan-April, 2003.

[4] A Nagarajan and S. N. Krishnan, "Continuous Monotonic Decomposition of Some Special Class of Graphs," *International Journal of Management and Systems,* Vol. 21, No.1, pp. 91-106, Jan-Apr, 2005.

[5] D. M. Burton, Elementary Number Theory, New Delhi: Universal Book Stall, 1998.

[6] J. Varghese and A. Antonysamy, "On the Continuous Monotonic Decomposition of Some Complete Tripartite Graphs," *Mapana Journal of Sciences,* Vol 8, No. 2, pp. 7-19, July-December, 2009.

[7] J. Varghese and A. Antonysamy, "On Double Continuous Monotonic Decomposition of Graphs," *Journal of Computer and Mathematical Sciences,* Vol. 1, No. 2, pp. 217-222, 2010.

[8] J. Varghese and A. Antonysamy, "On Modified Continuous Monotonic Decomposition of Graphs," *International Journal of Contemporary Mathematical Sciences,* Vol. 5, No. 33, pp. 1609-1614, 2010.