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Non-Uniform Leaf Springs with Geometric Nonlinearity

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***Abstract* - The present study deals with the analysis of large deflection of cantilever beams for various cross sections with a transverse load at free end. The aim of this analysis is to study the vertical and horizontal displacement behaviour of leaf springs which is traditionally modeled as cantilever beams of variable cross section. Besides the free end displacement, the variation of stress, strain and the bending moment of the beam are obtained by the technique of minimization of total potential energy principle. The displacement functions are approximated by linear combination of sets of orthogonal coordinate functions, developed through Gram-Schmidt scheme and substituted in the governing equilibrium equation. The final solution of the large displacement geometric nonlinear problem is obtained iteratively with the help of matlab computaional simulation. It is observed that the free end displacements and the maximum stress at fixed end are greatly affected by the geometry of the beam cross section. The present computational method has been validated and some new results have been furnished.**

***Keywords* : Geometric Nonlinearity, Leaf Spring, Cantilever Beam, Variable Cross Section**

I. Introduction

Leaf springs are the simplest form of spring, commonly made out of flat plates and used for the suspension in vehicles. The advantage of leaf spring over helical spring is that, they carry lateral loads, brake torque, driving torque etc, in addition to shocks. Multi-leaf springs consist series of flat plates, which are held together by means of two U-bolts and a centre clip. The suspension characteristics of such springs are highly nonlinear and leaf springs theory aims to find the optimized shapes of the leaves.

Deflections and stresses for non-linear bending are discussed and compared with those of a traditional leaf spring by Rajendran and Vijayarangan [1]. Design and manufacture of automotive leaf spring using functionally graded and composite materials have been addressed by several researchers [2, 3]. Sugiyama *et al* [4] reported development of nonlinear elastic leaf spring model for multi body vehicle systems. Rahman *et al* [5] carried out non-linear geometric analysis of parabolic leaf spring and extended the study for inelastic deformations [6].

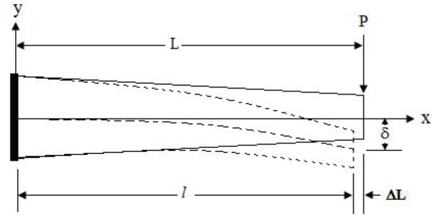


Fig. 1 Deflected shape of the beam

The geometrically nonlinear large deflection problem of cantilever beam had been solved classically by Bisshopp and Drucker [7]. Wang [8, 9] had proposed a numerical method for analyzing the nonlinear beam bending problem for concentrated and uniformly distributed load. Banerjee *et al.* [10] used analytical and numerical approaches to study large deflection of cantilever beams with geometric non-linearity by using non-linear shooting and adomian decomposition methods. Almeida *et al.* [11] used a tailored Lagrangian formulation for functionally graded cantilever beams of rectangular and hollow circular cross-section. In this paper, we investigate numerically a leaf spring system in which the cumulative effect of geometric nonlinearities are considered. The study undertakes a non-uniform cantilever with nonlinearity due to large deflection. The mathematical formulation is based on a variational method using total potential energy functional and solution is sought through Galerkin's assumed mode method.

II. Mathematical Formulation

Beam bending analysis based on the Bernoulli-Euler theory cannot be applied in the case of large deflections. Hence an iterative method using this simple beam theory has been implemented with appropriate length correction in each load step. Since the beams are quite slender for the present case, only pure bending is considered in this study ignoring the effect of shearing stresses. A schematic representation of bending curve for a cantilever beam with a point load P is shown in Fig. 1, along with its original configuration.

The governing differential equation of the beam system in question is derived by using minimization of total potential energy. Mathematically this is represented as $\delta(\pi) = 0$, where δ is the variational operator and π is the total potential energy. Noting that $\pi = \mathrm{U} + \mathrm{V}$, the equation can be expressed as,

$$\delta\pi = \delta(\mathrm{U} + \mathrm{V}) = 0 \quad (1)$$

The expression for strain energy of the beam is,

$$\mathrm{U} = \frac{1}{2} E \int_0^{\mathrm{L}} \mathrm{I_x} \left(\frac{\mathrm{d^2w}}{\mathrm{dx^2}}\right)^2 \mathrm{dx} \quad (2)$$

The expression of the potential energy, arising from the work done by the external uniform (p) and concentrated (P) load at the tip of the cantilever beam, is given below.

$$\mathrm{V} = -P\mathrm{w}_{|\mathrm{x=L}} - \int_0^{\mathrm{L}} (p\mathrm{w})\mathrm{dx} \quad (3)$$

Substituting the expressions of U and V in Eq. (1), and after carrying out some mathematical manipulations, we get

$$\delta\left[\frac{E}{2L^3}\int_0^1 I_\xi\left(\frac{d^2w}{d\xi^2}\right)^2 d\xi - Pw|_{\xi=1} - L\int_0^1 pwd\xi\right] = 0, \quad (4)$$

where ξ is the normalized length coordinate ($\xi = x/L$).The displacement functions $w(\xi)$ in Eq. (4), can be approximated by a linear combination of sets of orthogonal coordinate functions as

$$w(\xi) \cong \sum C_i\phi_i \ , \quad i=1, 2, ..., n. \quad (5)$$

The set of orthogonal functions ϕ_i are developed through Gram-Schmidt scheme, in which a starting function is used to generate the higher order orthogonal functions. The starting function ϕ_0 necessary in the first hand, is selected by satisfying the boundary conditions at fixed and free end as given below.

$$w|_{x=0} = 0 \text{ and } \frac{dw}{dx}|_{x=0}$$

$$EI\frac{d^2w}{dx^2}|_{x=L} = 0 \text{ and } EI\frac{d^3w}{dx^3}|_{x=L} = 0 \quad (6)$$

Substituting Eq. (5) in Eq. (4) yields the solution for the unknown displacement field w.

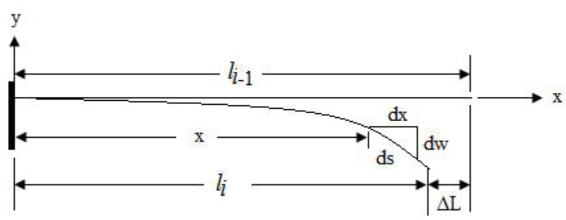


Fig. 2: Representation of bending curve for an intermediate load step

A. Effect of Stretching

The mathematical formulation as presented above is applicable for small deflection only $(\delta \cong t/L)$, beyond which strain-displacement relations become non-linear. In the present paper, the effect of geometric nonlinearity is implemented iteratively, in which the total load on the beam is imposed incrementally. In each load step, a correction on projected beam length is carried out such that the length of the deflection curve remains constant to the original straight length of the beam. This is shown through a schematic representation of the bending curve in Fig. 2. The elemental beam length shown in figure is given by $ds = \left(1+\left(\frac{dw}{dx}\right)^2\right)^{1/2} dx$ and this is integrated to obtain the stretching of beam length. $\Delta = \int_0^L\left((1+(dw/dx)^2)^{\frac{1}{2}}-1\right)dx = \int_0^L \frac{1}{2}\left(\frac{dw}{dx}\right)^2 dx$. The stretching Δ and deflection w produces horizontal (ΔL) and vertical (δ) components. ΔL is determined to obtain the effective beam length (l) for the next incremental load step, i.e., $(l_i = l_{i-1} - \Delta L)$. However, in each load step, analysis is carried out by following small deflection beam theory as presented in the previous section.

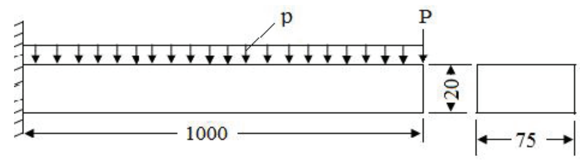


Fig. 3 Prismatic cantilever beam under combined loading

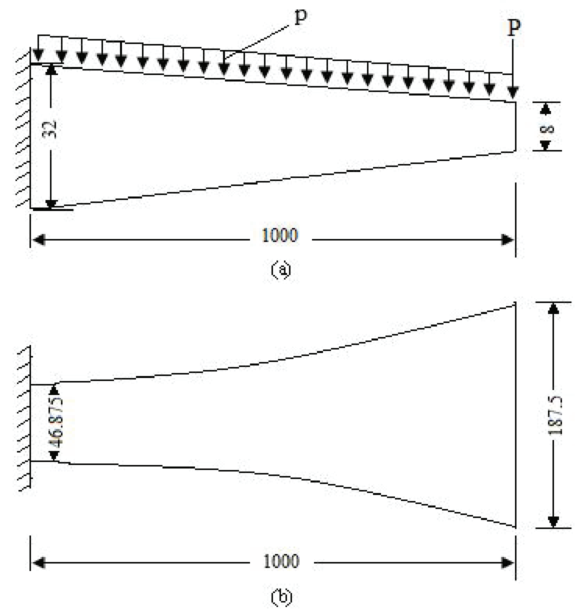


Fig. 4 Cantilever beam with linear variation in thickness (a) front view and (b) top view

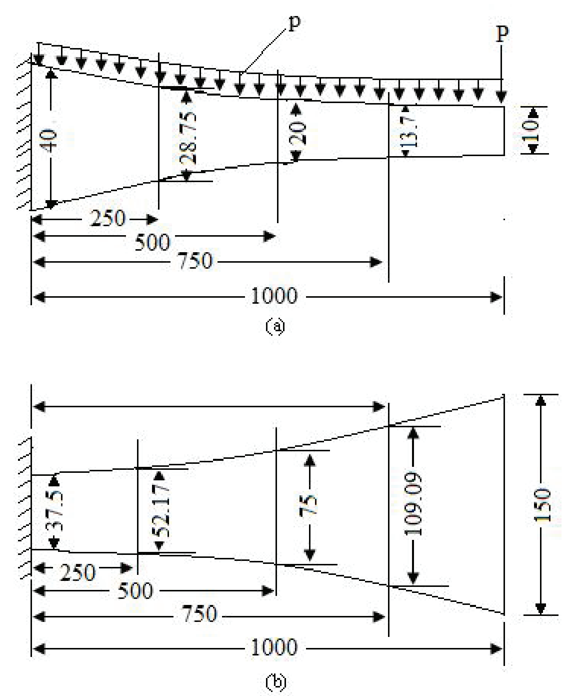


Fig. 5 Cantilever beam with quadratic variation in thickness (a) front view and (b) top view

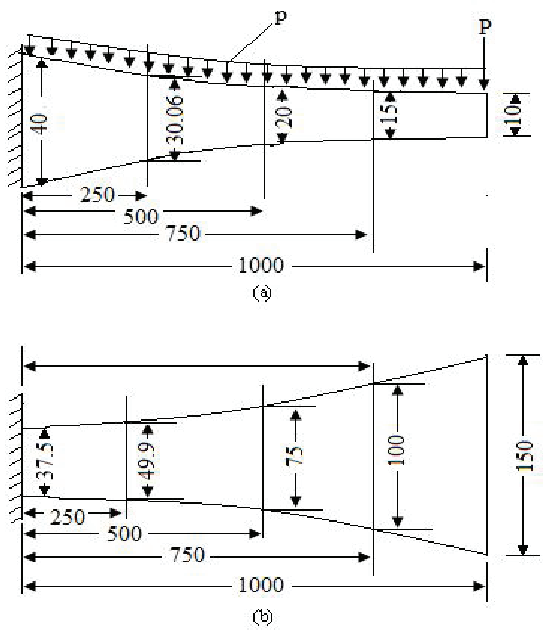


Fig. 6: Cantilever beam with fourth order variation in thickness (a) front view and (b) top view

III. Results And Discussions

The present study deals with the analysis of large deflection of cantilever beams for various rectangular cross sections under transverse load. The first configuration consists of a prismatic bar whereas in the second, thickness varies linearly and width is varied to maintain constant area of the cross section. The third and fourth configurations correspond to quadratic and fourth order variation in thickness with corresponding variation in width for constant cross-sectional area. The dimensional details of the different types of beams are shown in Figs. 3-6 (dimensions shown in mm).

TABLE I NUMERICAL VALUES OF LOADING CORRESPONDING TO LOAD PARAMETER L2(P+PL)/E I_M) =10.

Type of load	*P* in N	*p* in (N/m)	Increment of load
Concentrated	105000	0	45
Uniform distributed	0	105000	45
Combined	52500	52500	45

Validation of present results for a prismatic cantilever beam is carried out with the analytical results of Wang [8, 9] for concentrated and uniformly distributed load and shown in Fig. 7 (a, b). The agreements of the results establish the validity of the present method. Figure 7 (c) shows characteristic curve for prismatic beam under combined loading.

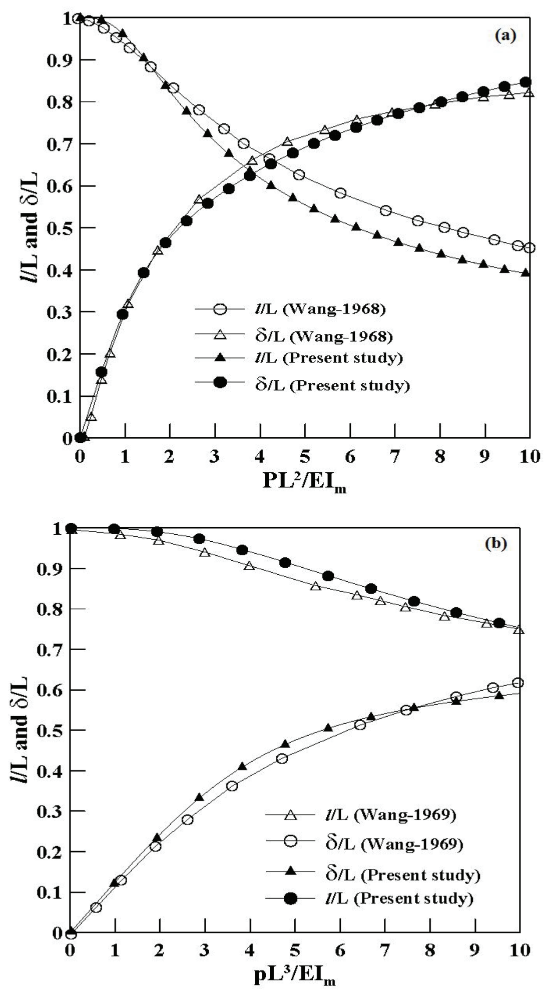
Variation of δ/L and l/L with PL2/E I_m, pL3/E I_m and L2(P+pL)/EI_m) for the concentrated, uniform and combined loads are shown in Figs. 8-10, for various configurations of beam cross section.

It is observed from the figures that changes in values of (l/L) and (δ/L) are more pronounced for beams with prismatic, linear, quadratic and fourth order variation in thickness. It should also be noted that the changes are positive (increasing) and negative (decreasing) respectively with (*l*/L) and (δ/L). The observation is highlighted through numerical values in table 2 for concentrated loading only but the trends remain same for all types of loadings.

The bending stress induced in the parabolic beam is calculated and validated with the results of Rahaman [5] but they are not furnished here to maintain brevity. The results for constant width beam, as they are commonly used in leaf springs, would be more meaningful from practical application view point and are also omitted in view of space limitation.

TABLE II VALUE OF (L/L) AND (δ/L) CORRESPONDING TO L2(P+PL)/E I_m) =10 FOR CONCENTRATED LOADING

Types of beam	(*l*/L)	(δ/L)
Prismatic	0.389361	0.848871
Linear variable	0.400830	0.47239
Quadratic variable	0.412970	0.390847
Fourth order	0.427200	0.379343



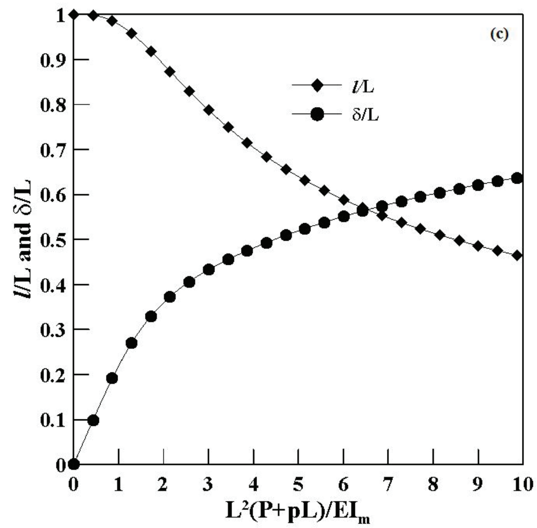
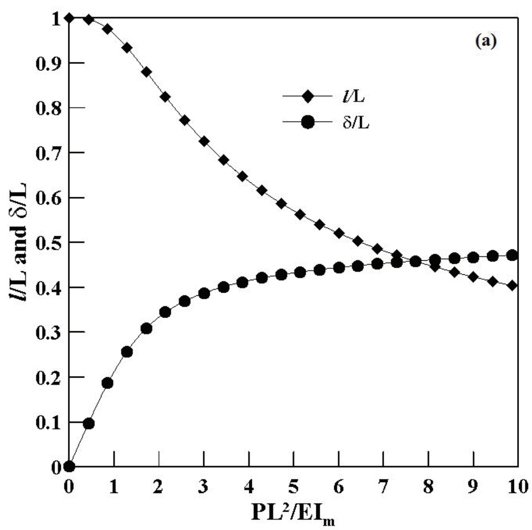
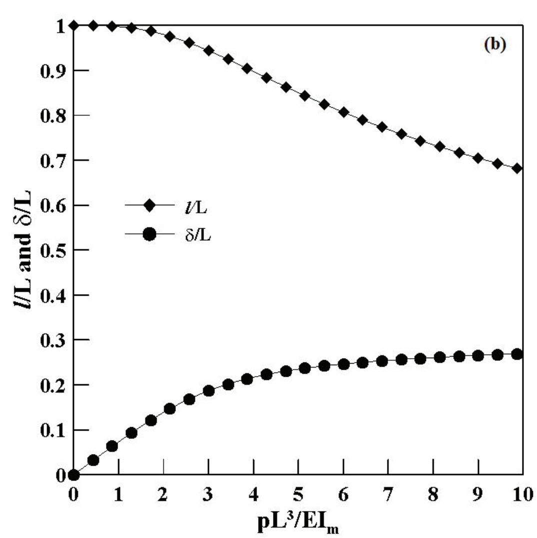


Fig. 7 Variation of δ/L and l/L with L2(P+pL)/E) for (a) concentrated load (b) uniformly distributed load and (c) under combined loading in prismatic cantilever beam. Validation of present results with Wang [8, 9] is also indicated in (a, b) for concentrated and uniformly distributed load





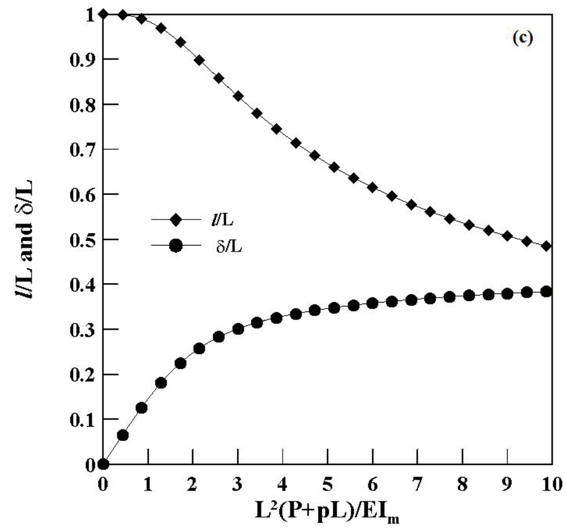
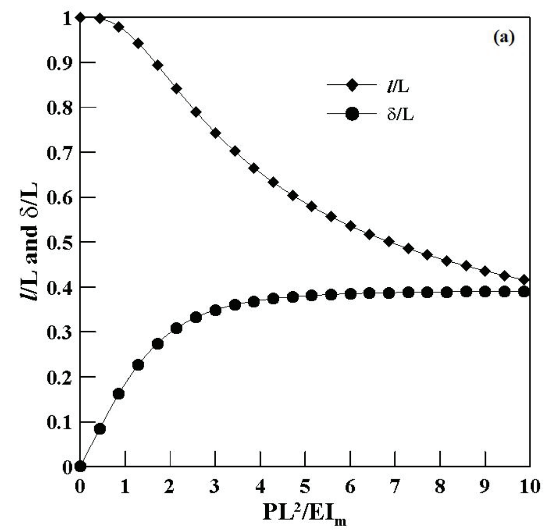
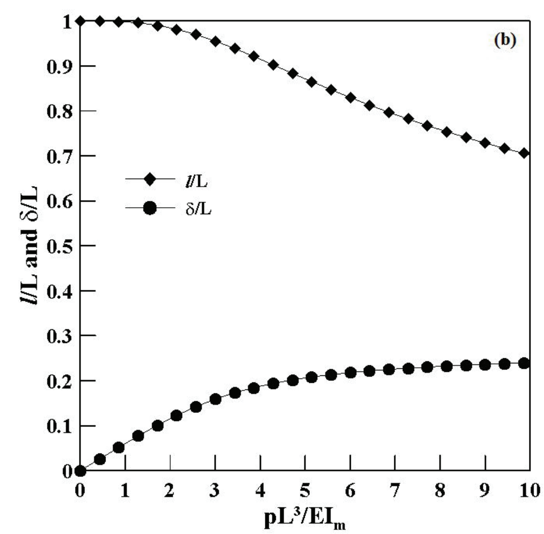


Fig. 8 Variation of δ/L and l/L with L2(P+pL)/E) for (a) concentrated load (b) uniformly distributed load and (c) under combined loading in linear cantilever beam





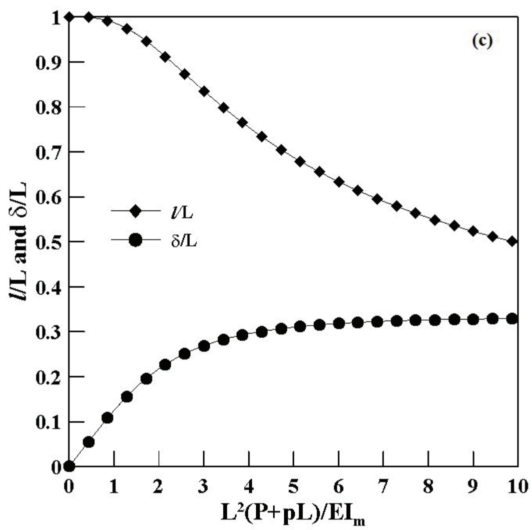
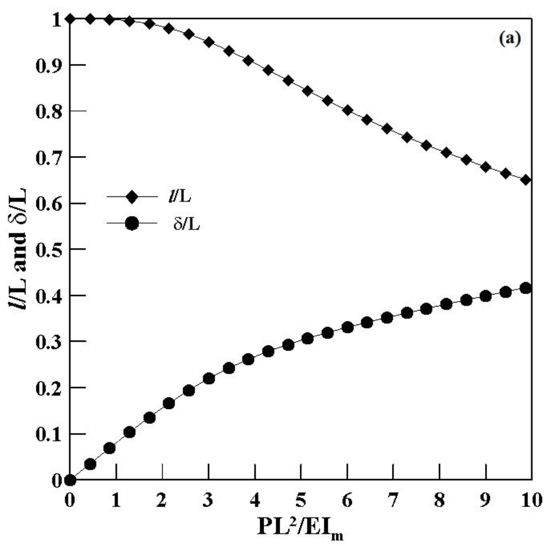
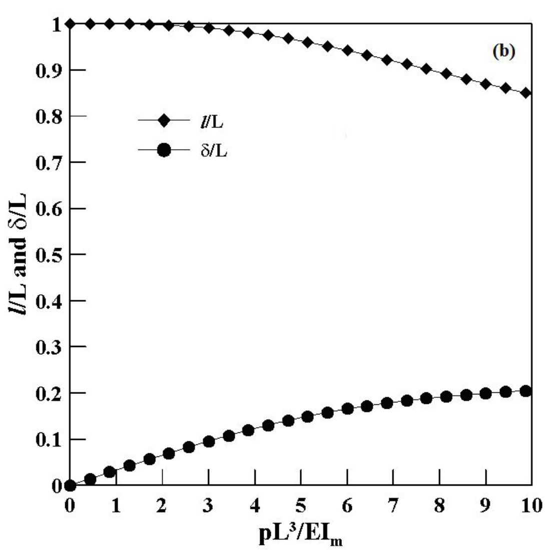


Fig. 9 Variation of δ/L and l/L with L2(P+pL)/E) for (a) concentrated load (b) uniformly distributed load and (c) under combined loading in quadratic cantilever beam





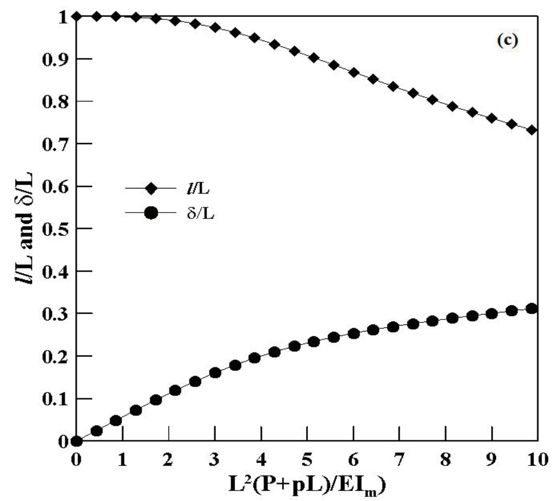


Fig. 10 Variation of δ/L and l/L with L2(P+pL)/E) for (a) concentrated load (b) uniformly distributed load and (c) under combined loading in fourth order cantilever beam

IV. Conclusion

An energy based variational method is proposed to analyze geometric nonlinearity of non-uniform leaf springs following the concept of updated Lagrangian analysis. The system is formulated as cantilever beam problem in which the effect of geometric nonlinearities are considered incrementally. The fundamental formulation is based on a variational method using total potential energy functional and solution is sought through Galerkin's assumed mode method. The final solution of the large displacement geometric nonlinear problem is obtained iteratively with the help of MATLAB computaional simulation. The present computational method has been successfully validated with existing results and some new results have been furnished. The present method, being based on an iterative computational technique, may be used to extend the problem in the area of thermo-elasticity and material non homogeneity.

Nomenclature

l	Current horizontal length of beam
c_i	Unknown coefficients
E	Elastic modulus of beam material
I, I_m	Area moment of inertia of beam section
[K]	Stiffness matrix
L	Length of beam
$\{C\}$	The vector of unknown coefficients
n	Numbers of orthogonal functions
P	Magnitude of concentrated load
p	Magnitude of uniform distributed load
t, t_x, t_m	Thickness of beam
U	Strain energy stored in the system
V	Potential energy of external forces
w	Transverse displacement field
ϵ	Strain
σ	Bending stress
ϕ_i	Set of orthogonal functions
ξ	Normalized axial coordinate
δ	Variational operator, Beam deflection

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